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# INTEGRATED CAUCHY FUNCTIONAL EQUATION WITH AN ERROR TERM AND THE EXPONENTIAL LAW

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*SUMMARY.* We characterize the positive solutions of the functional equation

$$f(x)(1-S(x)) = \int_0^\infty f(x+y)d\mu(y), \quad x \geqslant 0$$

where $|s(x)| \leqslant ce^{-\varepsilon x}$ and $\mu$ is a $\sigma$-finite positive Borel measure on $[0, \infty)$. The solutions are applide to study the stability of the characterizing properties of the integrated lack of memory property, conditional expectation and Pareto distribution.

## 1. Introduction

Many characterization problems in statistics can be reduced to determine the nonnegative, locally integrable solution $f$ of the following integrated Cauchy functional equation

$$f(x) = \int_0^\infty f(x+y)d\mu(y), \; x \geqslant 0$$

where $\mu$ is a positive $\sigma$-finite Borel measure (see e.g., Brandhofe and Davis, 1980; Davis, 1980; Galambos and Kotz, 1978; Lau and Rao, 1982; Ramachandran, 1979; Ramachandran and Rao, 1970; Shimizu, 1978). One of the most celebrated result of this kind is the Choquet-Deny theorem where $f$ is assumed to be bounded and $\mu$ is a probability measure : the solution is a periodic function with every $z$ in the support of $\mu$ as a period (Choquet and Deny, 1960). In particular if the support of $\mu$ is not a lattice, $f$ reduces to a constant. In the last decade, there are many literatures tried to remove the boundedness conditions on $f$ and on $\mu$ in the Choquet-Deny theorem (Brandhofe and Davis, 1980; Ramachandran, 1979; Ramachandran and Rao, 1970; Shimizu, 1980). It is, only recently, proved by Lau and Rao (1982)

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that such conditions are indeed redundant. The proof requires only elementary real analysis technique, and the method has been further simplified by Ramachandran (1982). Also about the same time, Davies (1980, 1982) pointed out that such a characterization has already been obtained by Deny (1960) in a more general setting, and has been overlooked for years. Deny's proof depends on the deeper Choquet theory.

In this paper, we are interested in the integrated Cauchy functional equation with an error term. Such problems has been studied by Shimizu (1980). However, his theorem contains some restrictive hypotheses on the solution $f$ and the measure $\mu$. We will settle this problem with some minimal assumptions. We call the equation

$$f(x)(1-S(x)) = \int_0^\infty f(x+y)d\mu(y), \quad x \geqslant 0$$

where $|S(x)| \leqslant Ce^{-\varepsilon x}$, the *integrated Cauchy functional equation with an error term*, and abbreviate by $\epsilon$-*ICFE*$(\mu)$. We prove that

Theorem : *Let $\mu$ be a positive $\sigma$-finite Borel measure with $\mu(0) < 1$, and let $f$ be a positive locally integrable solution of the $\epsilon$-ICFE$(\mu)$, then*

$$f(x) = p(x)e^{\alpha x}(1+K(x)), \quad x \geqslant 0$$

*where*

(i) *$\alpha$ is uniquely determined by* $\int_0^\infty e^{\alpha x}d\mu(x) = 1$;

(ii) *$p$ is a periodic function with every $z \in \operatorname{supp} \mu$ as a period, and*

(iii) *$K$ satisfies* $|K(x)| \leqslant C_1e^{-\varepsilon x}$ *for some* $x_0 \geqslant 0$, *and*

$$C_1 = 2C(1-\mu(0))^{-1}\left(1-\int_0^\infty e^{(\alpha-\varepsilon)x}d\mu(x)\right)^{-1}.$$

Shimizu (1980) has used this type of theorem to investigate the stability of the solutions arising from the lack of memory property and order statistics. We will apply the above theorem to study the lack of memory property again, the conditional expectation (Sahobov and Geshev, 1974), and the Pareto distributions.

## 2. Some lemmas

The main result of this section is Lemma 2.3.

Lemma 2.1 : *Let $f \geqslant 0$ be a locally integrable function on $[0, \infty)$ and let $h(x) = \int_0^1 f(x+y)dy$. Suppose $x_0 > 1$ and $h(x_0) > k > 0$, then there exists an interval $I$ containing $x_0$, with length $|I| \geqslant 1$ and $h(x) \geqslant \frac{1}{4}k$ for all $x \in I$.*

*Proof:* The function $h$ is continuous on $[0, \infty)$. Let $0 < \delta < 1$ satisfies $h(x_0-\delta) = \frac{1}{4}k$, and $h(x) \geqslant \frac{1}{4}k \;\forall\; x_0-\delta < x \leqslant x_0$ (if such $\delta$ does not exist, then the lemma is true). Then

$$\int_{-\delta}^{1-\delta} f(x_0+y)dy = \int_0^1 f(x_0-\delta+y)dy = h(x_0-\delta) = \frac{1}{4}k.$$

Since

$$\int_0^1 f(x_0+y)dy = h(x_0) \geqslant k,$$

it follows that

$$\int_{1-\delta}^{1} f(x_0+y)dy \geqslant \int_0^1 f(x_0+y)dy - \int_{-\delta}^{1-\delta} f(x_0+y)dy \geqslant \frac{3}{4}k.$$

Hence for $x_0 \leqslant x \leqslant x_0+(1-\delta)$,

$$h(x) = \int_0^1 f(x+y)dy = \int_{x-x_0}^{x-x_0+1} f(x_0+y)dy \geqslant \int_{1-\delta}^{1} f(x_0+y)dy \geqslant \frac{1}{4}k.$$

The interval $I = (x_0-\delta, x_0+1-\delta)$ satisfies the requirement.

Lemma 2.2: *Suppose* $f \geqslant 0$ *and satisfies the* $\varepsilon$-ICFE$(\mu)$. *Let* $h(x) = \int_0^1 f(x+y)dy$. *Then* $h$ *satisfies the* $\varepsilon$-ICFE$(\mu)$ *also, and is of bounded exponential order* (*i.e., there exist* $\alpha \in R$ *and* $k > 0$ *such that* $h(x) \leqslant ke^{\alpha x}$ *for all* $x \in [0, \infty)$).

*Proof:* By Fubini's theorem, the $\varepsilon$-ICFE$(\mu)$ reduces to

$$h(x) = \int_0^\infty h(x+y)d\mu(y) + \int_0^1 f(x+y)S(x+y)dy.$$

Let

$$S_0(x) = \begin{cases} \int_0^1 f(x+y)S(x+y)dy/h(x), & \text{if } h(x) \neq 0 \\ 0\qquad\qquad\qquad\qquad\qquad, & \text{if } h(x) = 0. \end{cases}$$

It is easy to show that with such definition, $h$ and $S_0$ satisfy

$$h(x) = \int_0^\infty h(x+y)d\mu(y) + h(x)S_0(x), \quad x \geqslant 0, \qquad \ldots (2.2)$$

and

$$|S_0(x)| \leqslant Ce^{-\varepsilon x}\int_0^1 f(x+y)e^{-\varepsilon y}dy/h(x) \leqslant Ce^{-\varepsilon x}.$$

We will prove that $h$ is bounded exponential order by showing that $\overline{\lim\limits_{x\to\infty}}\,(h(x))^{1/x} < \infty$. Suppose $\overline{\lim\limits_{x\to\infty}}\,(h(x))^{1/x} = \infty$. Let $a > 0$ be such that

$\mu(a, a+\frac{1}{2}) = b > 0$; we can assume, without loss of generality, that $a = 1$. Choose $k$ and $x_0$ such that

$$(h(x_0))^{1/x_0} > k > \frac{4(1+c)}{b}.$$

Hence $h(x_0) > k^{x_0}$ and by Lemma 2.1, we can assume that

$$h(x) > \frac{1}{4} k^{x_0} \;\forall\; x \in [x_0, x_0+1].$$

Let $x_1 = x_0 - 1$, then by (2.2),

$$h(x_1)(1-S_0(x_1)) \geqslant \int_1^{3/2} h(x_0-1+y)d\mu(y).$$

It follows that

$$h_1(x) \geqslant \frac{bk^{x_0}}{4(1-S_0(x_1))} \geqslant \frac{bk^{x_0}}{4(1+C)}.$$

Let $\gamma = \frac{bk}{4(1+C)} > 1$, then

$$h(x_1) \geqslant \gamma k^{x_0-1}$$

By applying Lemma 2.1 again, we can find an interval $(x_1-\delta_1, x_1+1-\delta_1)$ such that

$$h(x) \geqslant \frac{1}{4} \gamma k^{x_0-1} \;\forall\; x \in [x_1-\delta_1, x_1+1-\delta_1].$$

Let $x_2 = x_1-\delta_1-1$, the same argument as above yields

$$h(x_2) \geqslant \gamma^2 k^{x_0-2}.$$

We repeat this process to select the numbers. The process will terminate at some $m$, such $m$ and $x_m$ satisfy $\frac{x_0}{2} \leqslant m < [x_0]$, $0 \leqslant x_m \leqslant 2$, and

$$h(x_m) \geqslant \gamma^m k^{x_0-m} \geqslant \gamma^{x_0/2}.$$

Since $\gamma > 1$ and $x_0$ can be chosen arbitrary large, the function $h$ is unbounded on $[0, 2]$. This contradicts the continuity of $h$.

Lemma 2.3 : *Let $f \geqslant 0$ be a solution of the* ε-ICEF($\mu$). *Then*

(i) *There exists an $\alpha$ such that $e^{-\alpha x}h(x)$ is integrable.*

(ii) *If we let*

$$\tilde{f}(x) = \int_x^\infty e^{-\alpha y}h(y)dy, \quad d\tilde{\mu}(y) = e^{\alpha y}d\mu(y),$$

*then $\tilde{f}(x)$ is continuous, decreasing and satisfies*

$$\tilde{f}(x)(1-\tilde{S}(x)) = \int_0^\infty \tilde{f}(x+y)d\tilde{\mu}(y), \quad x \geqslant 0 \qquad \ldots \quad (2.3)$$

*where* $$|\tilde{S}(x)| \leqslant Ce^{-\varepsilon x}, \quad x \geqslant 0,$$

(iii) *Suppose* $\tilde{\mu}$ *satisfies* $\int_0^\infty e^{\lambda y} d\tilde{\mu}(y) > 1$ *and* $\tilde{\mu}(0) = 0$, *then there exists a* $k > 0$ *such that*

$$\tilde{f}(x) \leqslant ke^{\lambda x}.$$

*Proof:* The existence of the $\alpha$ in (i) follows from Lemma 2.2. Statement (ii) follows from a direct application of Fubini's theorem to equation (2.2).

To prove (iii), we will write $f$, $\mu$ and $S$ instead of $\tilde{f}$, $\tilde{\mu}$ and $\tilde{S}$ in order to simplify the notations (we only need the continuity and decreasing property of $f$). By multiplying a factor of $e^{\lambda' x}$ to $f(x)$, we may assume that $\lambda < 0$. The lemma will be proved if we can show that $\overline{\lim}_{x \to \infty} \frac{f(x)}{e^{\lambda x}} = 0$.

We first show that there exists a $\beta < \lambda$ such that $\underline{\lim}_{x \to \infty} \frac{f(x)}{e^{\beta x}} = 0$. For otherwise, $\underline{\lim}_{x \to \infty} \frac{f(x)}{e^{\beta x}} > 0$ for all $\beta < \lambda$, and hence $\underline{\lim}_{x \to \infty} \frac{f(x)}{e^{\beta x}} = \infty$ for all $\beta < \lambda$. We will choose $M$, $a > 0$ and $\beta < \lambda$ so that,

$$\int_a^M e^{\beta y} d\mu(y) = b > 1.$$

Let $x_0$ satisfy $b/(1+Ce^{-\varepsilon x_0}) > 1$, and for each $k$ let $x_k > x_0$ be such that

$$f(x) > ke^{\beta x} \;\forall\; x > x_k.$$

Now for $x > x_k$, $x-a > x_0$,

$$f(x-a)[1-S(x-a)] \geqslant \int_a^M f(x-a+y) d\mu(y) \geqslant ke^{\beta(x-a)} \int_a^M e^{\beta y} d\mu(y).$$

This implies that

$$f(x-a) \geqslant ke^{\beta(x-a)} b/(1+Ce^{-\varepsilon(x-a)}) \geqslant ke^{\beta(x-a)}.$$

Repeating the above argument we have, for $x > x_k$ with $x-na > x_0$,

$$f(x-na) \geqslant ke^{\beta(x-na)}. \qquad \ldots \quad (2.4)$$

In particular, if $x-na = x' \in [x_0, x_0+1]$, then

$$f(x') \geqslant ke^{\beta x'}.$$

Since $k$ is arbitrary, the continuity of $f$ on $[x_0, x_0+1]$ leads to contradiction.

To prove $\overline{\lim}_{x\to\infty} \frac{f(x)}{e^{\lambda x}} = 0$, we assume that $\overline{\lim}_{x\to\infty} \frac{f(x)}{e^{\lambda x}} > 0$. By the previous paragraph, we can find a, $M > 0$ and $\beta < \lambda$ such that

$$\overline{\lim_{x\to\infty}} \frac{f(x)}{e^{\beta x}} = \infty, \quad \underline{\lim_{x\to\infty}} \frac{f(x)}{e^{\beta x}} = 0 \qquad \dots \quad (2.5)$$

and

$$\int_a^M e^{\beta y} d\mu(y) > 1.$$

Let $x_0$ be defined the same as in last paragraph, and choose $x_k$, $y_k$ so that: $y_{k-1} < x_k < y_k$,

$$f(x_k) = e^{\beta x_k},\ f(y_k) > ke^{\beta y_k}$$

and

$$f(x) \geqslant e^{\beta x}\ \forall\ x_k \leqslant x \leqslant y_k.$$

(The existence of $x_k$ and $y_k$ are guaranteed by (2.5)). Since $f$ is decreasing, it follows that

$$k \leqslant f(y_k)e^{-\beta y_k} \leqslant f(x_k)e^{-\beta y_k} = e^{\beta x_k}\cdot e^{-\beta x_k} = e^{-\beta(y_k - x_k)},$$

and hence

$$y_k - x_k \geqslant \frac{\ln k}{-\beta} > 0$$

(the last inequality is a consequence of our previous assumption that $\beta < \lambda < 0$). We thus obtain a sequence of interval $\{[x_k, y_k]\}_{k=1}^{\infty}$ so that

(a) $\lim_{k\to\infty} (y_k - x_k) = \infty$, $\lim_{k\to\infty} x_k = \infty$;

(b) $f(x) \geqslant e^{\beta x}\ \forall\ x_k \leqslant x \leqslant y_k$.

Let $k$ be an integer and satisfies $y_k - x_k > M$, we can use the same technique as in obtaining (2.4) to show that

$$f(x) \geqslant e^{\beta x},\ x_0 \leqslant x \leqslant x_k.$$

Since $x_k$ and $y_k$ can be chosen arbitrary large, the above conclusion contradicts the choice of $\beta$ that $\underline{\lim}_{x\to\infty} \frac{f(x)}{e^{\beta x}} = 0$.

## 3. The main theorems

In the following theorem, the equation is slightly different from the ε-ICFE($\mu$), it partially generalizes (Shimizu, 1980, Theorem 3) by eliminating the redundant conditions on $f$ and $\mu$.

Theorem 3.1 : *Let $f$ be a nonnegative locally integrable function. Suppose $f$ satisfies*

$$f(x) = \int_0^\infty f(x+y)d\mu(y) + S(x), \quad x \geqslant 0 \qquad \dots \quad (3.1)$$

*where $\mu$ is a probability measure on $[0, \infty)$ with $\mu(0) < 1$, and $S(x)$ satisfies*

$$|S(x)| < Ce^{-\varepsilon x}.$$

*Then*
$$f(x) = p(x) + A(x)$$

*where $p(x)$ is a nonnegative periodic function with every $z \in supp\ \mu$ as a period and $A(x)$ is a real function bounded by $Ce^{-\varepsilon x}(1-\gamma)^{-1}$ with*

$$\gamma = \int_0^\infty e^{-\varepsilon y} d\mu(y).$$

*Proof:* Let $\mu^n$ denote the $n$-th convolution of $\mu$. It follows from (3.1) that

$$\begin{aligned} f(x) &= \int_0^\infty f(x+y)d\mu(y) + S(x) \\ &= \int_0^\infty \left( \int_0^\infty f(x+y+y_1)d\mu(y_1) + S(x+y) \right) d\mu(y) + S(x) \\ &= \int_0^\infty f(x+z)d\mu^2(z) + \int_0^\infty S(x+y)d\mu(y) + S(x) \\ &\vdots \\ &= \int_0^\infty f(x+z)d\mu^n(z) + \sum_{i=0}^{n-1} \int_0^\infty S(x+z)d\mu^i(z). \end{aligned}$$

Let
$$A(x) = \sum_{i=0}^\infty \int_0^\infty S(x+z)d\mu^i(z), \qquad \ldots \quad (3.2)$$

and let $\gamma = \int_0^\infty e^{-\varepsilon y}\, d\mu(y)$ $(< 1)$, then

$$\begin{aligned} \sum_{i=0}^\infty \int_0^\infty |S(x+z)|d\mu^i(z) &\leqslant Ce^{-\varepsilon x} \sum_{i=0}^\infty \int_0^\infty e^{-\varepsilon z}d\mu^i(z) \\ &= Ce^{-\varepsilon x} \sum_{i=0}^\infty \gamma^i \\ &= \frac{Ce^{-\varepsilon x}}{1-\gamma} < \infty, \end{aligned}$$

this implies $A(x)$ is finite for each $x \geqslant 0$ and $\lim_{n\to\infty} \int_0^\infty f(x+z)d\mu^n(z)$ exists a.e.

Define
$$g(x) = f(x) - A(x)$$

then
$$g(x) = \lim_{n\to\infty} \int_0^\infty f(x+z)d\mu^n(z) \quad \text{a.e., } x \geqslant 0,$$

and for almost all $x \geqslant 0$

$$\int_0^\infty g(x+y)d\mu(y) = \int_0^\infty f(x+y)d\mu(y) - \int_0^\infty A(x+y)d\mu(y)$$

$$= f(x) - S(x) - \sum_{i=0}^\infty \int_0^\infty S(x+z)d\mu^{i+1}(z)$$

$$= f(x) - \sum_{i=0}^\infty \int_0^\infty S(x+z)d\mu^i(z)$$

$$= f(x) - A(x)$$

$$= g(x).$$

It follows that $g$ is nonnegative and

$$g(x) = \int_0^\infty g(x+y)d\mu(y).$$

From Lau and Rao, 1982, Theorem 3.2, we have

$$g(x) = p(x)$$

where $p(x)$ is a nonnegative periodic function with $z \in \operatorname{supp} \mu$ as periods. Therefore

$$f(x) = p(x) + A(x)$$

and $A(x)$ satisfies
$$|A(x)| \leqslant \frac{Ce^{-\varepsilon x}}{1-\gamma}.$$

The next theorem on the solution of the ε-ICFE($\mu$), where $\mu$ is a $\sigma$-finite Borel measure on $[0, \infty)$, bears the same idea of proof as Theorem 3.1. However, because of the possible unboundedness of the term

$$\sum_{i=0}^\infty \int_0^\infty f(x+y)S(x+y)d\mu^i(y)$$

(compare (3.2) to (3.4)), we will need some elaborated proofs and the lemmas in the previous section.

Theorem 3.2 : *Suppose $\mu$ is a $\sigma$-finite Borel measure on $[0, \infty)$ with $\mu(0) < 1$. Suppose $f$ is a nonnegative, locally integrable solution of the* ε-ICFE($\mu$) *and $f \not\equiv 0$ on any interval $[a, \infty)$, $a > 0$. Then there exist $d_0 \in R$ and $x_0 > 0$ such that*

$$\int_0^\infty e^{\alpha_0 y}\, d\mu(y) = 1$$

*and for $x \geqslant x_0$, $f$ can be represented as*

$$f(x) = p(x)e^{\alpha_0 x}(1+K(x)),$$

*where $p(x)$ is a positive periodic function with every $z \in supp\ \mu$ as a period and*

$$|K(x)| < \frac{2C}{(1-\mu(0))(1-\gamma)}\ e^{-\varepsilon x}\ \forall\ x \geqslant x_0$$

*with*
$$\gamma = \int_0^\infty e^{(\alpha_0-\varepsilon)y}\ d\mu(y).$$

*In particular if $C$ satisfies $C < \frac{1}{2}(1-\mu(0))(1-\gamma)$, then $x_0$ can be chosen as zero.*

*Proof:* Without loss of generality, we assume that $\mu$ is nondegenerated at 0, and $\mu(0) = 0$. For otherwise, we let $\bar{\mu} = (\mu-\mu(0))/(1-\mu(0))$, then $\bar{\mu}(0) = 0$ and the ε-ICEF($\mu$) reduces to

$$f(x)\left(1-\frac{S(x)}{1-\mu(0)}\right) = \int_0^\infty f(x+y)d\bar{\mu}(y).$$

By the assumption that $f$ is positive and by Lemma 2.3 (iii), there exists a $\lambda_0$ such that

$$\int_0^\infty e^{\lambda_0 y} d\mu(y) < \infty.$$

This yields a $\lambda_1$ such that

$$\int_0^\infty e^{\lambda y} d\mu(y) \begin{cases} > 1, & \text{if } \lambda > \lambda_1, \\ < 1, & \text{if } \lambda < \lambda_1. \end{cases}$$

By readjusting the ε-ICEF($\mu$) with $e^{\lambda_1 x}$, we may take $\lambda_1 = 0$ and the above expression can be written as

$$\int_0^\infty e^{\lambda y} d\mu(y) \begin{cases} > 1, & \text{if } \lambda > 0, \\ < 1, & \text{if } \lambda < 0. \end{cases} \qquad \ldots \quad (3.3)$$

(Note that $\int_0^\infty d\mu(y)$ in this case is either 1 or $\infty$). We obtain from the ε-ICFE($\mu$) and the same proof as in the last theorem that

$$f(x) = \int_0^\infty f(x+z)d\mu^n(z) + \sum_{i=0}^{n-1} \int_0^\infty f(x+z)S(x+z)d\mu^i(z).$$

Let
$$A(x) = \sum_{i=0}^\infty \int_0^\infty f(x+z)S(x+z)d\mu^i(z). \qquad \ldots \quad (3.4)$$

We claim that $A(x)$ is finite a.e.: Let $\tilde{f}$ be defined as in Lemma 2.3 (ii), since

$$\int_0^\infty e^{\lambda y} d\tilde{\mu}(y) = \int_0^\infty e^{(\lambda+\alpha)y}\ d\mu(y).$$

We have (by (3.3)),

$$\int_0^\infty e^{\lambda y} d\tilde{\mu}(y) \begin{cases} > 1, & \text{if } \lambda > -\alpha, \\ < 1, & \text{if } \lambda < -\alpha. \end{cases}$$

If we let $\lambda = -\alpha + \frac{\varepsilon}{2}$, then Lemma 2.3 implies that there exists a $k > 0$ (may depend on $\varepsilon$) such that

$$\tilde{f}(x) < k e^{-(\alpha - \varepsilon/2)x}.$$

For any $x > 0$,

$$\begin{aligned} &\left| \int_x^\infty e^{-\alpha y} \int_0^1 A(y+t) dt dy \right| \\ &\leqslant C \sum_{i=0}^\infty \int_0^\infty e^{\alpha z} \left( \int_{x+z}^\infty e^{-\alpha y} \int_0^1 f(y+t) e^{-\varepsilon(y+t)} dt dy \right) d\mu^i(z) \\ &\leqslant C \sum_{i=0}^\infty \int_0^\infty e^{\alpha z - \varepsilon(x+z)} \tilde{f}(x+z) d\mu^i(z) \qquad \cdots \quad (3.5) \\ &\leqslant C k e^{-(\alpha+\varepsilon/2)x} \sum_{i=0}^\infty \int_0^\infty e^{-(\varepsilon/2)z} d\mu^i(z). \end{aligned}$$

So,

$$\left| \int_x^\infty e^{-\alpha y} \int_0^1 A(y+t) dt dy \right| \leqslant \frac{C k e^{-(\alpha+\varepsilon/2)x}}{1-\gamma_1} \qquad \cdots \quad (3.6)$$

where

$$\gamma_1 = \int_0^\infty e^{-(\varepsilon/2)y} d\mu(y).$$

It follows that $A(x)$ is finite a.e. and the proof of the claim is complete.

The function

$$g(x) = f(x) - A(x) = \lim_{n\to\infty} \int_0^\infty f(x+z) d\mu^n(z) \ (\geqslant 0)$$

is hence defined almost everywhere. We will show that

(i) $g(x) \not\equiv 0$ : For otherwise, $f(x) = A(x)$, it follows from the definition of $A(x)$ and (3.6) that

$$\tilde{f}(x) \leqslant \frac{C k e^{-(\alpha+\varepsilon/2)x}}{1-\gamma_1}. \qquad \cdots \quad (3.7)$$

Also, (3.5) implies that

$$\tilde{f}(x) \leqslant C \sum_{i=0}^\infty \int_0^\infty e^{\alpha z - \varepsilon(x+z)} \tilde{f}(x+z) d\mu^i(z).$$

Substitute (3.7) into the above inequality successively, we have

$$\tilde{f}(x) \leqslant k e^{-\alpha x} \left[ \frac{C e^{-(\varepsilon/2)x}}{1-\gamma_1} \right]^n.$$

Choose $x_0$ such that

$$\frac{C e^{-(\varepsilon/2)x_0}}{1-\gamma_1} < \tfrac{1}{2}. \qquad \cdots \quad (3.8)$$

and let $n \to \infty$, then $\tilde{f}(x) = 0$ for almost all $x \in [x_0, \infty)$. This implies $f(x) = 0$ for almost all $x \in [x_0, \infty)$, and contradicts that $f \not\equiv 0$ on $[a, \infty)$, for any $a > 0$.

(ii) $g$ satisfies $g(x) = \int_0^\infty g(x+y)d\mu(y)$ : from the proof of (3.6), we know that

$$\sum_{i=0}^{\infty} \int_0^\infty |f(x+z)S(x+z)| d\mu^i(z)$$

converges absolutely. By using the same argument as the counterpart in Theorem 3.1, we obtain the conclusion.

Now Theorem 3.2 in Lau and Rao (1982) implies that

$$g(x) = p(x)e^{\alpha_0 x}, \; x \geqslant x_0$$

where $x_0$ satisfies (3.2), $\alpha_0$ is uniquely determined by

$$\int_0^\infty e^{\alpha_0 y} \, d\mu(y) = 1,$$

and $p(x)$ is a positive periodic function with periods $z \in \operatorname{supp} \mu$. It follows from our assumption (3.4) on $\lambda_1 = 0$, that $\alpha_0 = \lambda_1 = 0$ and $\mu$ is actually a probability measure. Therefore

$$g(x) = p(x)$$

and

$$f(x) = p(x)+A(x). \qquad \ldots \quad (3.9)$$

It remains to estimate the term $A(x)$. Substituting $f(x)$ in the definition of $A(x)$ in (3.4) by the expression (3.9), we have

$$A(x) = p(x)A_1(x)+B_1(x), \quad x \geqslant 0, \qquad \ldots \quad (3.10)$$

where

$$A_1(x) = \sum_{i=0}^{\infty} \int_0^\infty S(x+z)d\mu^i(z),$$

and

$$B_1(x) = \sum_{i=0}^{\infty} \int_0^\infty A(x+z)S(x+z)d\mu^i(z).$$

Since

$$|A_1(x)| \leqslant \sum_{i=0}^{\infty} \int_0^\infty |S(x+z)| d\mu^i(z) \leqslant Ce^{-\varepsilon x} \sum_{i=0}^{\infty} \int_0^\infty e^{-\varepsilon z} \, d\mu^i(z)$$

and $\mu$ is a probability measure,

$$|A_1(x)| \leqslant Ce^{-\varepsilon x} \sum_{i=0}^{\infty} \gamma^i = \frac{Ce^{-\varepsilon x}}{1-\gamma}$$

where $\gamma = \int_0^\infty e^{-\varepsilon y} \, d\mu(y)$. By substituting the expression of $A(x)$ in (3.10) to $B_1(x)$, we have

$$B_1(x) = p(x)A_2(x)+B_2(x)$$

where $$A_2(x) = \sum_{i=0}^{\infty} \int_0^{\infty} A_1(x+z)S(x+z)d\mu^i(z)$$

and $$B_2(x) = \sum_{i=0}^{\infty} \int_0^{\infty} B_1(x+z)S(x+z)d\mu^i(z).$$

The same argument as above yields

$$|A_2(x)| < \left(\frac{Ce^{-\varepsilon x}}{1-\gamma}\right)^2.$$

Inductively, we have

$$A_n(x) = p(x)\left(\sum_{i=1}^{n} A_i(x)\right) + B_n(x) \qquad \dots \quad (3.11)$$

and $$|A_n(x)| \leqslant \left(\frac{Ce^{-\varepsilon x}}{1-\gamma}\right)^n.$$

Let $$K(x) = \sum_{i=1}^{\infty} A_i(x).$$

Then for $x \geqslant x_0$ ($x_0$ is determined in (3.8))

$$|K(x)| \leqslant Ce^{-\varepsilon x}\left(\frac{1-\gamma}{1-\gamma-Ce^{-\varepsilon x}}\right) \leqslant \frac{2Ce^{-\varepsilon x}}{1-\gamma}.$$

If we can show that $\lim_{n\to\infty} B_n(x) = 0$ for $x \geqslant x_0$, then

$$f(x) = p(x)(1+K(x)), \quad x \geqslant x_0 \qquad \dots \quad (3.12)$$

and the proof of the theorem will be complete. By using the identity

$$\int_x^{\infty} e^{-\alpha y} \int_0^1 B_1(y+t)dtdy = \sum_{i=0}^{\infty} \int_0^{\infty} \int_x^{\infty} e^{-\alpha y} \int_0^1 f(y+z+t)S(y+z+t)dtdyd\mu^i(z)$$

and by applying the same proof as for (3.6) with

$$|A(x)| \leqslant \frac{Ce^{-\varepsilon x}}{1-\gamma},$$

we have $$\int_x^{\infty} e^{-\alpha y} \int_0^1 |B_1(y+t)|dtdy \leqslant ke^{-\alpha x}\left(\frac{Ce^{-\varepsilon x}}{1-\gamma}\right)^2 < \infty.$$

Inductively, we have

$$\int_x^{\infty} e^{\alpha y} \int_0^1 |B_n(y+t)|dtdy \leqslant ke^{-\alpha x}\left(\frac{Ce^{-\varepsilon x}}{1-\gamma}\right)^{n+1}.$$

Therefore $$\lim_{n\to\infty} \int_{x_0}^{\infty} e^{-\alpha y} \int_0^1 |B_n(y+t)|dtdy = 0,$$

which is equivalent to $\{B_n\}$ converges to 0 in measure. Since $K(x) = \sum_{i=1}^{\infty} A_i(x)$ exists a.e., the relation (3.11) implies that

$$\lim_{n\to\infty} B_n(x) = 0 \quad \text{a.e., } x \geqslant x_0.$$

To prove the particular case in the theorem, we note that the choice of $x_0$ in (3.8) depends only on the number $\lambda = -\alpha + \frac{\varepsilon}{2}$. Now that we already know the explicit form of $f$, we can show that

$$\tilde{f}(x) \leqslant ke^{-\alpha x}, \quad x \geqslant 0$$

for some $k > 0$. The $\varepsilon/2$ can actually be taken as 0 and hence the $x_0$ can be taken as

$$\frac{Ce^{-\varepsilon x_0}}{1-\gamma} < \tfrac{1}{2}$$

where $\gamma = \int_0^\infty e^{-\varepsilon x} d\mu(x)$. The last statement follows immediately from this observation.

Corollary 3.3 : *Suppose $\mu(0) < 1$ and suppose the constant $C$ in the error term of the $\varepsilon$-ICFE($\mu$) satisfies*

$$C < \tfrac{1}{2}(1-\mu(0))(1-\gamma)$$

*where $\gamma = \int_0^\infty e^{(\alpha-\varepsilon)x} d\mu(y)$ and $\int_0^\infty e^{\alpha x} d\mu(y) = 1$. If $f$ is a nonnegative, locally integrable solution of the $\varepsilon$-ICFE($\mu$), then $f$ can be represented as*

$$f(x) = p(x)e^{\alpha x}(1+K(x)) \quad \forall\, x \geqslant 0$$

*where $p$, and $K$ satisfy the same conditions as in Theorem 3.2.*

If $\int_0^\infty e^{\alpha x} d\mu(x) \neq 1$ for any $\alpha \in R$, it can be shown that $f = 0$ a.e.

If $\mu$ is a measure such that $\mu(0) > 1$, then the $\varepsilon$-ICFE($\mu$) can be written as

$$f(x)(1-\mu(0)-S(x)) = \int_{(0,\infty)} f(x+y)d\mu(y), \quad x \geqslant 0.$$

The right side is positive, and the left side is negative for large value of $x$. This implies that $f \equiv 0$ for $x \geqslant x_0$ for some $x_0 > 0$.

We do not have a conclusion for the case $\mu(0) = 1$.

## 4. Some applications

Throughout this section, we will consider the non-lattice random variables only, the lattice random variables can be handled similarly.

A nonnegative random variable $X$ is said to have the *lack of memory property* if

$$P(X > x+y) = P(X > x)P(X > y) \ \forall\, x, y \in [0, \infty).$$

It is well known that such $X$ follows the exponential law. Recently, there are two extensions of the above property (Lau and Rao, 1982; Ramachandran, 1979; Shimizu, 1978). Shimizu (1980, Theorem 6) has included the error term to one of the extensions. We will reproduce his theorem here. For a random variable $X$, we will let $F$ denote its distribution and let $G = 1-F$.

Theorem 4.1 : *Let $X$, $Y$ be two independent random variables with distributions $F_1$ and $F_2$, and $F_2(0+) = 0$. Suppose $X$, $Y$ satisfy*

$$P(X > x+y/X > Y) = P(X > x)(1-S(x)) \ \forall\, x \geqslant 0 \qquad \ldots \quad (4.1)$$

*where* $|S(x)| \leqslant Ce^{-\varepsilon x}$, *and*

$$C < \frac{\delta}{6}\left[1-\left(P(X > Y)^{-1}.\int_0^\infty e^{-(\alpha+\varepsilon)y}dF_2(y)\right)\right]$$

*with $\delta > 0$ given, and* $P(X > Y) = \int_0^\infty e^{-\alpha y}dF_2(y)$. *Then*

$$F_1(x) = 1-e^{\alpha x}(1+K_1(x)) \ \forall\, x \geqslant 0$$

*and* $|K_1(x)| \leqslant \delta$.

*Proof :* The identify (4.1) can be reduced to

$$\int_0^\infty G_1(x+y)dF_2^*(y) = G_1(x)(1-S(x)), \quad x \geqslant 0$$

where $F_2^*(y) = F_2(y)/P(X > Y)$. By applying Corollary 3.3, we have

$$G_1(x) = pe^{-\alpha x}(1+K(x)) \ \forall\, x \geqslant 0$$

where $|K(x)| \leqslant \frac{\delta}{3}e^{-\varepsilon x}$. Note that $p.(1+K(0)) = 1$ and $|K(0)| < \frac{\delta}{3}$, by expressing $G_1$ as

$$G_1(x) = e^{-\alpha x}(1+K_1(x)) \ \forall\, x \geqslant 0$$

where $K_1(x) = (p-1)+pK(x)$, we obtain the conclusion.

Sahobov and Geshev (1974) established that a nonnegative random variable $X$ which satisfies

$$E((X-z)^k/X \geqslant z) = E(X) \ \forall\, z \geqslant 0$$

is an exponential distribution.

Theorem 4.2 : *Let $X$ be a random variable with distribution $F$, and satisfies* $F(0+) = 0$,

$$E((X-z)^k/X \geqslant z) = E(X^k)(1-S(z)) \ \forall\, z \geqslant 0 \qquad \ldots \quad (4.2)$$

*where* $|S(z)| \leqslant Ce^{-\varepsilon z}$, *and*

$$C < \frac{\delta}{6}\left[1-\left(E(X^k)^{-1}.\int_0^\infty e^{-(\alpha+\varepsilon)y}d(y^k)\right)\right]$$

*with* $\delta > 0$ *given, and* $E(X^k) = \int_0^\infty e^{-\lambda y} d(y^k)$. *Then*

$$F(x) = 1 - e^{-\lambda x}(1 + K_1(x)) \ \forall \ x \geqslant 0$$

*where* $|K_1(x)| \leqslant \delta$.

*Proof:* Equation (4.2) can be reduced to

$$\int_0^\infty G(y+z) y^{k-1} dy = \frac{E(X^k)}{k} G(z)(1 - S(z)) \ \forall \ z \geqslant 0$$

and by using the same argument as above the theorem follows.

The Pareto distribution is defined by

$$F(x) = \begin{cases} 1 - a^k x^{-k}, & x \geqslant a, \\ 0 \quad , & x < a. \end{cases}$$

It is easily seen that the transformation $x = e^y$ makes $y$ an exponential variable. The Pareto law plays an important role in the study of income distribution (Krishnaji, 1970; Lau and Rao, 1982).

Theorem 4.3 : *Let* $X$ *be a nonnegative random variable* $X$ *truncated at* $a > 0$, *and let* $R$ *be an independent random variable over the interval* (0, 1). *If the distribution of* $Y = XR$ *satisfies*

$$P(Y > x) = P(X > x)(1 - S(x)), \ x \geqslant 0 \qquad \dots \ (4.3)$$

*where* $|S(x)| < \frac{C}{x^\varepsilon}$ *and* $C$ *satisfies* $C < \frac{\delta a^\varepsilon}{6}\left(1 - \int_0^1 y^\varepsilon dH(y)\right)$ ($H$ *is the distribution of* $R$). *Then the distribution function* $F$ *of* $X$ *is of the form*

$$F(x) = \begin{cases} 1 - \frac{a^k}{x^k}(1 + K_1(x)), & x \geqslant a \\ 0 \quad , & x < a \end{cases}$$

*where* $|K_1(x)| \leqslant \delta$, $x \geqslant a$.

*Proof:* Equation (4.3) can be reduced to

$$\int_0^1 G\left(\frac{x}{r}\right) dH(r) = G(x)(1 - S(x)), \ \forall \ x \geqslant 0.$$

Let $x = e^u$, $r = e^{-v}$, $G_1(u) = G(e^u)$, $S_1(u) = S(e^u)$ and $H_1(v) = H(e^{-v})$. Then the above equation reduces to

$$\int_0^\infty G_1(u+v) dH_1(v) = G_1(u)(1 - S_1(u)), \quad \forall \ u \geqslant 0$$

with $|S_1(u)| \leqslant Ce^{-\varepsilon u}$. We can apply Corollary 3.3 to obtain the conclusion.

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